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Post-Newtonian expansion of gravitational waves from a particle in slightly eccentric orbit around a rotating black hole

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Abstract

Using the Teukolsky and Sasaki-Nakamura equations, we calculate the post-Newtonian expansion of the energy and angular momentum luminosities of gravitational waves from a test particle with slightly eccentric orbit around a rotating black hole up to $O(v^5)$ beyond the quadrupole formula. We find that circular orbit is stable up through the order $O(v^5)$ even if the spin of a black hole exists. Using the results, we argue the total phase of gravitational waves from coalescing compact binaries. We find that the eccentricity is important at $O(v^5)$ order if the initial eccentricity in the LIGO's band is greater than 0.4. There are possibilities that binaries which have a non-negligible eccentricity will be observed by laser interferometers and our results will be useful to analyze gravitational waves from such binaries.

1 Introduction

Among the possible sources of gravitational waves, coalescing compact binaries are the most promising candidates which can be detected by the near-future, ground based laser interferometric gravitational wave detectors such as LIGO [1] and VIRGO [2]. One reason is that we expect such events to occur 3/yr within 200Mpc [3]. The other reason is that we expect sufficiently large amplitude of gravitational waves to be detected by LIGO and VIRGO if such events occur.

We can extract physical information of binaries, such as masses and spins etc., from *inspiral wave forms* by the matched filtering technique [4]. In this method, binaries' parameters are determined by cross-correlating the noisy signal from detectors with theoretical templates. If the signal and the templates lose phase with each other by one cycle over the thousands as the waves sweep through the LIGO/VIRGO band, their cross correlation will be significantly reduced. This means that it is important to construct theoretical templates which are accurate to better than one cycle during entire sweep through the LIGO/VIRGO band [4]. If we have accurate templates, we can determine the chirp mass of the systems within 1% error [5]. Thus, much effort has been recently made to construct accurate theoretical templates [6–11].

To calculate inspiraling wave forms from coalescing binaries, the standard method is the post-Newtonian expansion of the Einstein equations, in which the equations are expanded in terms of a small parameter $v \sim (M/r)^{1/2}$, where M and r are the total mass and orbital length-scale of the system, respectively. However, calculations have been successful to only a few orders in v beyond the leading (Newtonian) order so far.

Recently the post-Newtonian expansion based on the perturbation formalism of a black hole is developed. In such analysis, we consider gravitational waves from a particle of mass μ orbiting around a black hole of mass M assuming $\mu \ll M$. Although this method is restricted to the case of $\mu \ll M$, we can calculate higher order post-Newtonian corrections of gravitational waves by means of relatively simple analysis compared to the standard post-Newtonian analysis. This direction of research was first done analytically by Poisson [12] to $O(v^3)$ order and numerically by Cutler et. al. [13] to $O(v^6)$ order. Then a highly accurate numerical calculation was done by Tagoshi and Nakamura [14] to $O(v^8)$ order and they found the appearance of $\log v$ terms in the energy flux at $O(v^6)$ and at $O(v^8)$. They also clarified that the accuracy of the energy flux to at least $O(v^6)$ is needed to construct template wave forms. An analytical calculation to the same order was done by Tagoshi and Sasaki [15] which was based on the formulation built up by Sasaki [16] and they confirmed the result of Tagoshi and Nakamura. These calculation were extended to a rotating black hole case by Shibata et.al. [17] to $O(v^5)$ order. They calculated gravitational waves from a particle in circular orbit with small inclination from the equatorial plane to see the effect of spin at higher post-Newtonian orders. They found that the effect of spin to the orbital phase is important at $O(v^5)$ order when one of the star is a rapidly rotating neutron star with its pulse period less than 2 ms or a rapidly rotating black hole with $q = J_{BH}/M^2 \geq 0.2$.

In this paper, we extend the analysis of Shibata et.al. to a case of eccentric orbit. The post-Newtonian expansion of gravitational waves including the effect of the eccentricity have already been done to $O(v^3)$ order for arbitrary eccentricity by Blanchet and Schäfer

[10] and the spin effect at $O(v^3)$ order was calculated by Shibata [18].

The binary pulsars such as PSR 1913+16 have now non-negligible eccentricity, but when gravitational waves from such systems can be detected by LIGO/VIRGO, their eccentricity will have significantly decreased due to radiation reaction and it will become negligible. However there are some possibilities that close binary systems are produced in dense stellar systems [19] which may evolve to super massive black holes in the center of AGN's. Those binaries may have highly eccentric orbit when they emit the gravitational waves which can be detected by LIGO/VIRGO. Since LIGO's advanced detector systems have sensitivities for BH-BH inspirals at < 5 Gpc. such binaries seem to be possible sources for LIGO/VIRGO. Then to construct the template for such binaries, it seems important to investigate the effect of eccentricity at higher post-Newtonian order. Hence, in this paper, we study the effect of eccentricity to $O(v^5)$ order including spin effect.

On the other hand, a laser interferometric detector in space, LISA, was proposed[20]. LISA will observe the gravitational waves which have amplitude $h \sim 10^{-23}$ at frequency $f \sim 10^{-1} - 10^{-3}$ by one year integration. Then gravitational waves from compact stars orbiting around supermassive black holes in galactic centers will become a possible sources for such detectors and a few events per year can be expected [18]. Then it is important to investigate the gravitational waves from super massive black holes. Since $\mu/M \ll 1$ for such cases, our formulas give the correct value under the assumption $v \ll 1$.

The paper is organized as follows. In section 2, we show the Teukolsky equation and review the method of the post-Newtonian expansion of a homogeneous solution of the Teukolsky equation for a Kerr black hole using the Sasaki-Nakamura equation. In section 3, we solve the geodesic equation assuming that eccentricity e of a particle is small ($e \ll 1$), and integrate the source term of the Teukolsky equation. In section 4, we show the energy and angular momentum luminosity to $O(e^2)$ and $O(v^5)$. In section 5, we study the stability of circular orbit under radiation reaction. In section 6, we study the effect of the eccentricity to orbital phase of coalescing binaries. Section 7 is devoted to the summary. Throughout this paper we use the units of $c = G = 1$.

2 General formulation

2.1 The Teukolsky equation

We consider the case when a test particle of mass μ travels a slightly eccentric orbit around a Kerr black hole of mass $M \gg \mu$. We follow the notation used by Shibata et.al., but for definiteness, we recapitulate necessary formulae and definitions of symbols.

To calculate gravitational radiation from a particle orbiting around a Kerr black hole, we start with the Teukolsky equation. [21, 22] We focus on the radiation going out to infinity described by the fourth Newman-Penrose quantity, ψ_4 [23], which may be expressed as

$$\psi_4 = (r - ia\cos\theta)^{-4} \int d\omega e^{-i\omega t} \sum_{\ell,m} \frac{e^{im\varphi}}{\sqrt{2\pi}} \,{}_{-2}S^{a\omega}_{\ell m}(\theta) R_{\ell m\omega}(r), \tag{2.1}$$

where ${}_{-2}S^{a\omega}_{\ell m}$ is the spheroidal harmonic function of spin weight $s = -2$, which are normalized as

$$\int_0^\pi |_{-2}S^{a\omega}_{\ell m}|^2 \sin\theta d\theta = 1. \tag{2.2}$$

The radial function $R_{\ell m\omega}(r)$ obeys the Teukolsky equation with spin weight $s = -2$,

$$\Delta^2 \frac{d}{dr}\Big(\frac{1}{\Delta}\frac{dR_{\ell m\omega}}{dr}\Big) - V(r)R_{\ell m\omega} = T_{\ell m\omega}(r), \tag{2.3}$$

where $T_{\ell m\omega}(r)$ is the source term whose explicit form will be shown later, and $\Delta = r^2 - 2Mr + a^2$. The potential $V(r)$ is given by

$$V(r) = -\frac{K^2 + 4i(r-M)K}{\Delta} + 8i\omega r + \lambda, \tag{2.4}$$

where $K = (r^2 + a^2)\omega - ma$ and λ is the eigenvalue of $_{-2}S^{a\omega}_{\ell m}$.

The solution of the Teukolsky equation at infinity ($r \to \infty$) is expressed as

$$R_{\ell m\omega}(r) \to \frac{r^3 e^{i\omega r^*}}{2i\omega B^{in}_{\ell m\omega}} \int_{r_+}^{\infty} dr' \frac{T_{\ell m\omega}(r')R^{in}_{\ell m\omega}(r')}{\Delta^2(r')} \equiv \tilde{Z}_{\ell m\omega} r^3 e^{i\omega r^*}, \tag{2.5}$$

where $r_+ = M + \sqrt{M^2 - a^2}$ denotes the radius of the event horizon and $R^{in}_{\ell m\omega}$ is the homogeneous solution which satisfies the ingoing-wave boundary condition at horizon,

$$R^{in}_{\ell m\omega} \to \begin{cases} D_{\ell m\omega}\Delta^2 e^{-ikr^*} & \text{for } r^* \to -\infty \\ r^3 B^{out}_{\ell m\omega} e^{i\omega r^*} + r^{-1} B^{in}_{\ell m\omega} e^{-i\omega r^*} & \text{for } r^* \to +\infty, \end{cases} \tag{2.6}$$

where $k = \omega - ma/2Mr_+$ and r^* is the tortoise coordinate defined by

$$\frac{dr^*}{dr} = \frac{r^2 + a^2}{\Delta}. \tag{2.7}$$

For definiteness, we fix the integration constant such that r^* is given explicitly by

$$\begin{aligned} r^* &= \int \frac{dr^*}{dr} dr \\ &= r + \frac{2Mr_+}{r_+ - r_-} \ln \frac{r - r_+}{2M} - \frac{2Mr_-}{r_+ - r_-} \ln \frac{r - r_-}{2M}, \end{aligned} \tag{2.8}$$

where $r_\pm = M \pm \sqrt{M^2 - a^2}$.

2.2 Post-Newtonian expansion of the homogeneous solution

In the previous papers [16, 17], the post-Newtonian expansion of the homogeneous solution was performed to $O(\epsilon^2)$ where $\epsilon \equiv 2M\omega$. In this section, we review the method [16, 17].

In order to calculate gravitational waves emitted to infinity from a particle in slightly eccentric orbit, we need to know the explicit form of the source term $T_{\ell m\omega}(r)$, which has support around $r = r_0$ where r_0 is the radius which is the value of order of the orbital radius in the Boyer-Lindquist coordinate, the ingoing-wave Teukolsky function $R^{in}_{\ell m\omega}(r)$ around $r = r_0$, and its incident amplitude $B^{in}_{\ell m\omega}$ at infinity. We consider the expansion of these quantities in terms of a small parameter. $v^2 \equiv M/r_0$. In addition, we need to expand those quantity in terms of $\epsilon \equiv 2M\omega$ since $\omega = O(\Omega)$ where Ω is the orbital angular velocity of the particle and $M\omega = O(v^3)$. In case of Kerr black hole, the another combination of parameters $a\omega$ appears in the Teukolsky equation. We define $a \equiv qM$ and we have $a\omega = q\epsilon/2 = O(\epsilon^3)$.

First we perform the expansion of the spheroidal harmonics ${}_{-2}S^{a\omega}_{\ell m}$ and their eigenvalues λ in terms of $a\omega$. Since the spheroidal harmonics will appear only in the source term and since $a\omega = O(\epsilon)$, we have only to evaluate ${}_{-2}S^{a\omega}_{\ell m}$ up to $O(a\omega)$. On the other hand, the angular eigenvalues λ comes into play in the radial equation, we evaluate it up to $O(a^2\omega^2)$. The spheroidal harmonics ${}_{-2}S^{a\omega}_{\ell m}$ is given by

$$ {}_{-2}S^{a\omega}_{\ell m} = {}_{-2}P_{\ell m} + a\omega S^{(1)}_{\ell m} + O((a\omega)^2), \tag{2.9} $$

where ${}_{-2}P_{\ell m}$ are the spherical harmonics of spin weight $s = -2$ [24] and

$$ S^{(1)}_{\ell m} = \sum_{\ell'} c^{\ell'}_{\ell m} \; {}_{-2}P_{\ell' m}\,, \tag{2.10} $$

where $c^{\ell'}_{\ell m}$ are non-zero only for $\ell' = \ell \pm 1$ and they are given by

$$ \begin{aligned} c^{\ell+1}_{\ell m} &= \frac{2}{(\ell+1)^2}\Big[\frac{(\ell+3)(\ell-1)(\ell+m+1)(\ell-m+1)}{(2\ell+1)(2\ell+3)}\Big]^{1/2}, \\ c^{\ell-1}_{\ell m} &= -\frac{2}{\ell^2}\Big[\frac{(\ell+2)(\ell-2)(\ell+m)(\ell-m)}{(2\ell+1)(2\ell-1)}\Big]^{1/2}. \end{aligned} $$

The eigenvalue λ is given by

$$ \lambda = \lambda_0 + a\omega\lambda_1 + a^2\omega^2\lambda_2 + O((a\omega)^3) \tag{2.11} $$

where $\lambda_0 = (\ell-1)(\ell+2)$, $\lambda_1 = -2m(\ell^2+\ell+4)/(\ell^2+\ell)$ and

$$ \lambda_2 = -2(\ell+1)(c^{\ell+1}_{\ell m})^2 + 2\ell(c^{\ell-1}_{\ell m})^2 + \frac{2}{3} - \frac{2}{3}\frac{(\ell+4)(\ell-3)(\ell^2+\ell-3m^2)}{\ell(\ell+1)(2\ell+3)(2\ell-1)}\,. \tag{2.12} $$

Next we solve the homogeneous solution $R^{in}_{\ell m\omega}$. The Teukolsky equation is transformed into Sasaki-Nakamura equation [25], which is given by

$$ \Big[\frac{d^2}{dr^{*2}} - F(r)\frac{d}{dr^*} - U(r)\Big]X_{\ell m\omega} = 0. \tag{2.13} $$

The explicit forms of $F(r)$ and $U(r)$ are given in the Appendix A. The relation between $R_{\ell m\omega}$ and $X_{\ell m\omega}$ is

$$ R_{\ell m\omega} = \frac{1}{\eta}\Big\{\Big(\alpha + \frac{\beta_{,r}}{\Delta}\Big)\chi_{\ell m\omega} - \frac{\beta}{\Delta}\chi_{\ell m\omega,r}\Big\}, \tag{2.14} $$

where $\chi_{\ell m\omega} = X_{\ell m\omega}\Delta/(r^2+a^2)^{1/2}$, and the functions α, β and η are shown in Appendix A. Conversely, we can express $X_{\ell m\omega}$ in terms of $R_{\ell m\omega}$ as

$$ X_{\ell m\omega} = (r^2+a^2)^{1/2} r^2 J_- J_- \Big[\frac{1}{r^2}R_{\ell m\omega}\Big], \tag{2.15} $$

where $J_- = (d/dr) - i(K/\Delta)$. Then the asymptotic behavior of the ingoing-wave solution $X^{in}_{\ell m\omega}$ which corresponds to Eq.(2.6) is

$$X^{in}_{\ell m\omega} \rightarrow \begin{cases} A^{out}_{\ell m\omega} e^{i\omega r^*} + A^{in}_{\ell m\omega} e^{-i\omega r^*} & \text{for } r^* \rightarrow \infty \\ C_{\ell m\omega} e^{-ikr^*}, & \text{for } r^* \rightarrow -\infty, \end{cases} \tag{2.16}$$

where $A^{in}_{\ell m\omega}$, $A^{out}_{\ell m\omega}$ and $C_{\ell m\omega}$ are respectively related to $B^{in}_{\ell m\omega}$, $B^{out}_{\ell m\omega}$ and $D_{\ell m\omega}$ defined in Eq.(2.6) as

$$\begin{aligned} B^{in}_{\ell m\omega} &= -\frac{1}{4\omega^2} A^{in}_{\ell m\omega}, \\ B^{out}_{\ell m\omega} &= -\frac{4\omega^2}{c_0} A^{out}_{\ell m\omega}, \\ D_{\ell m\omega} &= \frac{1}{d_{\ell m\omega}} C_{\ell m\omega}, \end{aligned} \tag{2.17}$$

where c_0 is given in Eq.(A.3) of Appendix A and

$$\begin{aligned} d_{\ell m\omega} &= \sqrt{2Mr_+}[(8 - 24iM\omega - 16M^2\omega^2)r_+^2 \\ &\quad + (12iam - 16M + 16amM\omega + 24iM^2\omega)r_+ - 4a^2m^2 - 12iamM + 8M^2]. \end{aligned}$$

First we introduce the variable $z = \omega r$ and

$$\begin{aligned} z^* &= z + \epsilon \left[\frac{z_+}{z_+ - z_-} \ln(z - z_+) - \frac{z_-}{z_+ - z_-} \ln(z - z_-) \right] \\ &= \omega r^* + \epsilon \ln \epsilon, \end{aligned} \tag{2.18}$$

where $z_\pm = \omega r_\pm$. To solve $X^{in}_{\ell m\omega}$ by expanding it in terms of ϵ, we set

$$X^{in}_{\ell m\omega} = \sqrt{z^2 + a^2\omega^2} \xi_{\ell m}(z) \exp(-i\phi(z)), \tag{2.19}$$

where

$$\begin{aligned} \phi(z) &= \int dr \left(\frac{K}{\Delta} - \omega \right) \\ &= z^* - z - \frac{\epsilon}{2} mq \frac{1}{z_+ - z_-} \ln \frac{z - z_+}{z - z_-}, \end{aligned} \tag{2.20}$$

which generalizes the phase function $\omega(r^* - r)$ of the Schwarzschild case. This prescription makes it easy to implement the ingoing-wave boundary condition on $X^{in}_{\ell m\omega}$.

Inserting Eq.(2.19) into Eq.(2.13) and expanding it in terms of $\epsilon = 2M\omega$, we obtain

$$L^{(0)}[\xi_{\ell m}] = \epsilon L^{(1)}[\xi_{\ell m}] + \epsilon Q^{(1)}[\xi_{\ell m}] + \epsilon^2 Q^{(2)}[\xi_{\ell m}] + O(\epsilon^3), \tag{2.21}$$

where $L^{(0)}$, $L^{(1)}$, $Q^{(1)}$ and $Q^{(2)}$ are differential operators given by

$$L^{(0)} = \frac{d^2}{dz^2} + \frac{2}{z}\frac{d}{dz} + \left(1 - \frac{\ell(\ell+1)}{z^2}\right), \tag{2.22}$$

$$L^{(1)} = \frac{1}{z}\frac{d^2}{dz^2} + \left(\frac{1}{z^2} + \frac{2i}{z}\right)\frac{d}{dz} - \left(\frac{4}{z^3} - \frac{i}{z^2} + \frac{1}{z}\right), \tag{2.23}$$

$$Q^{(1)} = \frac{iq\lambda_1}{2z^2}\frac{d}{dz} - \frac{4imq}{l(l+1)z^3} + \frac{4mq}{l(l+1)z^2} + \frac{\lambda_1 q + 2mq}{2z^2}, \tag{2.24}$$

for arbitrary ℓ and

$$\begin{aligned} Q^{(2)} &= -\frac{q^2}{4z^2}\frac{d^2}{dz^2} - \left(\frac{9q^2 - 2m^2q^2 - 6imq}{36z^3} + \frac{i(2m^2q^2 - 9q^2) - 18mq}{108z^2}\right)\frac{d}{dz} \\ &+ \left(\frac{9q^2 + 5m^2q^2 - 3imq}{9z^4} + \frac{i(2m^2q^2 - 9q^2)}{54z^3}\right. \\ &\left.- \frac{-63imq + 81q^2 - 2m^2q^2}{378z^2}\right), \end{aligned}$$

for $\ell = 2$. We do not need $Q^{(2)}$ for $\ell \geq 3$ for the post-Newtonian order we consider in this paper (i.e., up to $O(v^5)$ beyond Newtonian). Expanding $\xi_{\ell m}$ in terms of ϵ as

$$\xi_{\ell m} = \sum_{n=0}^{\infty} \epsilon^n \xi_{\ell m}^{(n)}(z), \tag{2.25}$$

we obtain from Eq.(2.21) the iterative equations,

$$L^{(0)}[\xi_{\ell m}^{(0)}] = 0, \tag{2.26}$$

$$L^{(0)}[\xi_{\ell m}^{(1)}] = L^{(1)}[\xi_{\ell m}^{(0)}] + Q^{(1)}[\xi_{\ell m}^{(0)}] \equiv W_{\ell m}^{(1)}, \tag{2.27}$$

$$L^{(0)}[\xi_{\ell m}^{(2)}] = L^{(1)}[\xi_{\ell m}^{(1)}] + Q^{(1)}[\xi_{\ell m}^{(1)}] + Q^{(2)}[\xi_{\ell m}^{(1)}] \equiv W_{\ell m}^{(2)}. \tag{2.28}$$

The general solution to Eq.(2.26) is immediately obtained as

$$\xi_{\ell m}^{(0)} = \alpha_\ell^{(0)} j_\ell + \beta_\ell^{(0)} n_\ell, \tag{2.29}$$

where j_ℓ and n_ℓ are usual spherical Bessel functions. The boundary condition for $n \leq 2$ is that $\xi_{\ell m}^{(n)}$ is regular at $z = 0$. Hence $\beta_\ell^{(0)} = 0$ and we set $\alpha_\ell^{(0)} = 1$ for convenience.

To calculate $\xi_{\ell m}^{(n)}$ for $n \geq 1$, we rewrite Eq.(2.27) and (2.28) in the indefinite integral form by using the spherical Bessel functions as

$$\xi_{\ell m}^{(n)} = n_\ell \int^z dz z^2 j_\ell W_{\ell m}^{(n)} - j_\ell \int^z dz z^2 n_\ell W_{\ell m}^{(n)} \quad (n = 1, 2). \tag{2.30}$$

For $n = 1$, we have

$$\begin{aligned} \xi_{\ell m}^{(1)} &= \alpha_\ell^{(1)} j_\ell + \frac{(\ell-1)(\ell+3)}{2(\ell+1)(2\ell+1)} j_{\ell+1} - \frac{\ell^2 - 4}{2\ell(2\ell+1)} j_{\ell-1} \\ &+ z^2(n_\ell j_0 - j_\ell n_0) j_0 + \sum_{k=1}^{\ell-1}\left(\frac{1}{k} + \frac{1}{k+1}\right) z^2(n_\ell j_k - j_\ell n_k) j_k \end{aligned}$$

$$+ \quad n_\ell(\mathrm{Ci}(2z) - \gamma - \ln 2z) - j_\ell \mathrm{Si}(2z) + i j_\ell \ln z$$
$$+ \quad \frac{imq}{2}\left(\frac{\ell^2+4}{\ell^2(2\ell+1)}\right) j_{\ell-1} + \frac{imq}{2}\left(\frac{(\ell+1)^2+4}{(\ell+1)^2(2\ell+1)}\right) j_{\ell+1}, \tag{2.31}$$

where $\mathrm{Ci}(x) = -\int_x^\infty dt \cos t/t$ and $\mathrm{Si}(x) = \int_0^x dt \sin t/t$ are cosine and sine integral functions, γ is the Euler constant, and $\alpha_\ell^{(1)}$ is an integration constant which represents the arbitrariness of the normalization of $X_{\ell m\omega}^{in}$. We set $\alpha_\ell^{(1)} = 0$ for simplicity.

As noted previously, the source term $T_{\ell m\omega}$ has support only around $r = r_0$ and $\omega r_0 = O(\Omega r_0) = O(v)$. Hence we only need $X_{\ell m\omega}^{in}$ at $z = O(v) \ll 1$ to evaluate the source integral, apart from the value of the incident amplitude $A_{\ell m\omega}^{in}$. Hence the post-Newtonian expansion of $X_{\ell m\omega}^{in}$ corresponds to the expansion not only in terms of $\epsilon = O(v^3)$ but also z by assuming $\epsilon \ll z \ll 1$. In order to evaluate the gravitational wave luminosity to $O(v^5)$ beyond the leading order, we must calculate the series expansion of $\xi_{\ell m}^{(n)}$ in powers of z for $n = 0$ to $\ell = 4$, for $n = 1$ to $\ell = 3$ and for $n = 2$ to $\ell = 2$.

When we evaluate $A_{\ell m\omega}^{in}$, we examine the asymptotic behavior of $\xi_{\ell m}^{(n)}$ at infinity. Since the accuracy of $A_{\ell m\omega}^{in}$ we need is $O(\epsilon)$, we don't have to calculate $\xi_{\ell m}^{(2)}$ in analytic form. We need only the series expansion formulas for $\xi_{\ell m}^{(2)}$ around $z = 0$, which is easily obtained by Eq.(2.30). Inserting $\xi_{\ell m}^{(n)}$ into Eq.(2.19) and expanding it by z and ϵ assuming $\epsilon \ll z \ll 1$, we obtain

$$\begin{aligned} X_{2m\omega}^{in} &= \frac{z^3}{15} - \frac{z^5}{210} + \frac{z^7}{7560} + O(z^9) + \epsilon\Big(\frac{imqz^2}{30} - \frac{13z^4}{630} - \frac{11imqz^4}{3780} + O(z^6)\Big) \\ &+ \epsilon^2\Big(\frac{q^2+2imq-m^2q^2}{120}z - \frac{mq}{30}z^2 + O(z^3)\Big), \end{aligned} \tag{2.32}$$

$$X_{3m\omega}^{in} = \frac{z^4}{105} - \frac{z^6}{1890} + O(z^8) + \epsilon\Big(-\frac{z^3}{126} + \frac{2imq}{945}z^3 + O(z^5)\Big), \tag{2.33}$$

$$X_{4m\omega}^{in} = \frac{z^5}{945} + O(z^7). \tag{2.34}$$

Using the transformation of Eq.(2.14), we obtain $R_{\ell m\omega}^{in}$ as

$$\begin{aligned} \omega R_{2m\omega}^{in} &= \frac{z^4}{30} + \frac{i}{45}z^5 - \frac{11\,z^6}{1260} - \frac{i}{420}z^7 + \frac{23\,z^8}{45360} + \frac{i}{11340}z^9 \\ &+ \epsilon\Big(\frac{-z^3}{15} - \frac{i}{60}m\,q\,z^3 - \frac{i}{60}z^4 + \frac{m\,q\,z^4}{45} \\ &- \frac{41\,z^5}{3780} + \frac{277\,i}{22680}m\,q\,z^5 - \frac{31\,i}{3780}z^6 - \frac{7\,m\,q\,z^6}{1620}\Big) \\ &+ \epsilon^2\Big(\frac{z^2}{30} + \frac{i}{40}m\,q\,z^2 + \frac{q^2\,z^2}{60} - \frac{m^2\,q^2\,z^2}{240} \\ &- \frac{i}{60}z^3 - \frac{m\,q\,z^3}{30} + \frac{i}{90}q^2\,z^3 - \frac{i}{120}m^2\,q^2\,z^3\Big), \end{aligned} \tag{2.35}$$

$$\begin{aligned} \omega R_{3m\omega}^{in} &= \frac{z^5}{630} + \frac{i}{1260}z^6 - \frac{z^7}{3780} - \frac{i}{16200}z^8 \\ &+ \epsilon\Big(\frac{-z^4}{252} - \frac{i}{1890}m\,q\,z^4 - \frac{i}{756}z^5 + \frac{11\,m\,q\,z^5}{22680}\Big), \end{aligned} \tag{2.36}$$

$$\omega R^{in}_{4m\omega} = \frac{z^6}{11340} + \frac{iz^7}{28350}. \tag{2.37}$$

We calculate $A^{in}_{\ell m\omega}$ by examining the asymptotic behavior of $\xi^{(n)}_{\ell m}$ and Eq.(2.19) at infinity. The results are expressed as

$$A^{in}_{\ell m\omega} = \frac{1}{2} i^{\ell+1} e^{-i\epsilon \ln \epsilon}\left[1 + \epsilon\left(-\frac{\pi}{2}\text{sgn}(\omega) + iq^{(1)}_{\ell m}\right) + \ldots\right], \tag{2.38}$$

where

$$q^{(1)}_{\ell m} = \frac{1}{2}\left[\psi(\ell) + \psi(\ell+1) + \frac{(\ell-1)(\ell+3)}{\ell(\ell+1)}\right] - \ln 2 - \frac{2imq}{\ell^2(\ell+1)^2}, \tag{2.39}$$

$$\psi(\ell) = \sum_{k=1}^{\ell-1}\frac{1}{k} - \gamma. \tag{2.40}$$

The corresponding incident amplitudes $B^{in}_{\ell m\omega}$ for the Teukolsky function are obtained from Eq.(2.17).

3 Source terms

3.1 The geodesic equations

In this section, we solve the geodesic equation for a slightly eccentric motion. The same procedure was previously performed by Apostolatos et.al. [26] for Schwarzschild geometry.

The geodesic equations in Kerr geometry are given by

$$\begin{aligned}
\Sigma\frac{d\theta}{d\tau} &= \pm\Big[C - \cos^2\theta\Big\{a^2(1-E^2) + \frac{l_z^2}{\sin^2\theta}\Big\}\Big]^{1/2} \equiv \Theta(\theta),\\
\Sigma\frac{d\varphi}{d\tau} &= -\Big(aE - \frac{l_z}{\sin^2\theta}\Big) + \frac{a}{\Delta}\Big(E(r^2+a^2) - al_z\Big) \equiv \Phi,\\
\Sigma\frac{dt}{d\tau} &= -\Big(aE - \frac{l_z}{\sin^2\theta}\Big)a\sin^2\theta + \frac{r^2+a^2}{\Delta}\Big(E(r^2+a^2) - al_z\Big) \equiv T,\\
\Sigma\frac{dr}{d\tau} &= \pm\sqrt{R},
\end{aligned} \tag{3.1}$$

where E, l_z and C are the energy, the z-component of the angular momentum and the Carter constant of a test particle, respectively.[1] $\Sigma = r^2 + a^2\cos^2\theta$ and

$$R = [E(r^2+a^2) - al_z]^2 - \Delta[(Ea - l_z)^2 + r^2 + C]. \tag{3.2}$$

We consider the motion of a particle in the equatorial plane $\theta = \pi/2$. In this case we can set $C = 0$. We define $r = r_0$ as the radius where the potential R/r^4 becomes the

[1] In this section, these constants of motion are those measured in units of μ. That is, if expressed in the standard units, E, l_z and C in Eq.(3.1) are to be replaced with E/μ, l_z/μ and C/μ^2, respectively.

minimum $\partial(R/r^4)/\partial r \mid_{r=r_0} = 0$. We also define the eccentricity e so that $r = r_0(1+e)$ is a turning point of the radial motion, at which $R(r = r_0(1+e)) = 0$. In this paper we assume $e \ll 1$. Using above two equations and expanding E and l_z as

$$
\begin{aligned}
E &= E^{(0)} + eE^{(1)} + e^2E^{(2)} + e^3E^{(3)} + O(e^4), \\
l_z &= l_z^{(0)} + el_z^{(1)} + e^2l_z^{(2)} + e^3l_z^{(3)} + O(e^4),
\end{aligned}
$$

we obtain $E^{(n)}$ and $l_z^{(n)}$ for $n = 0, 1$ and 2 as

$$
\begin{aligned}
E^{(0)} &= \frac{1-2v^2+qv^3}{(1-3v^2+2qv^3)^{(1/2)}}, \\
E^{(1)} &= 0, \\
E^{(2)} &= \frac{v^2(1-3v^2+qv^3+q^2v^4)(-1+6v^2-8qv^3+3q^2v^4)}{2(1-3v^2+2qv^3)^{3/2}(-1+2v^2-q^2v^4)}, \\
l_z^{(0)} &= \frac{r_0 v(1-2qv^3+q^2v^4)}{(1-3v^2+2qv^3)^{(1/2)}}, \\
l_z^{(1)} &= 0, \\
l_z^{(2)} &= \frac{q\, r_0\, v^5(q-3v+qv^2+q^2v^3)(-1+6v^2-8qv^3+3q^2v^4)}{2(1-3v^2+2qv^3)^{3/2}(-1+2v^2-q^2v^4)},
\end{aligned}
$$

The post-Newtonian expansion forms of $E^{(n)}$ and $l_z^{(n)}$ up to the required orders are

$$
E = 1 - \frac{M}{2\,r_0} + \frac{3\,M^2}{8\,{r_0}^2} - \frac{q\,M^{\frac{5}{2}}}{{r_0}^{\frac{5}{2}}} + e^2\left(\frac{M}{2\,r_0} - \frac{5\,M^2}{4\,{r_0}^2} + \frac{3\,q\,M^{\frac{5}{2}}}{{r_0}^{\frac{5}{2}}}\right) + O(v^6), \tag{3.3}
$$

$$
\begin{aligned}
l_z &= (Mr_0)^{1/2}\left[1 + \frac{3M}{2r_0} - \frac{3qM^{3/2}}{r_0^{3/2}} + \left(\frac{27}{8}+q^2\right)\frac{M^2}{r_0^2} - \frac{15qM^{5/2}}{2r_0^{5/2}}\right. \\
&\quad \left. + e^2\left(\frac{q^2M^2}{2r_0^2} - \frac{3qM^{5/2}}{2r_0^{5/2}}\right) + O(v^6)\right].
\end{aligned} \tag{3.4}
$$

Next we solve the geodesic equations for slightly eccentric orbits. The radial equation is

$$
\left(\frac{dr}{dt}\right)^2 = \frac{R}{T^2}. \tag{3.5}
$$

We expand $r(t)$ as

$$
r(t) = r_0\left[1 + er^{(1)}(t) + e^2r^{(2)}(t) + O(e^3)\right], \tag{3.6}
$$

and R/T^2 in terms of e and v using Eq.(3.3) and (3.4). Collecting terms of equal order in e, we obtain

$$
\left(\frac{dr^{(1)}}{dt}\right)^2 = \Omega_r^2(1-(r^{(1)})^2), \tag{3.7}
$$

and

$$\frac{1}{\Omega_r^2}\frac{dr^{(1)}}{dt}\frac{dr^{(2)}}{dt} = -r^{(1)}r^{(2)} + q_0 + q_1 r^{(1)} + q_2 (r^{(1)})^2. \tag{3.8}$$

In this paper, since we don't need the exact expression of Ω_r, q_0, q_1 and q_2, we only show its post-Newtonian forms:

$$\Omega_r = \frac{M^{1/2}}{r_0^{3/2}}\left[1 - \frac{3M}{r_0} + \frac{3qM^{3/2}}{r_0^{3/2}} - \frac{(9+3q^2)M^2}{2r_0^2} + \frac{15qM^{5/2}}{r_0^{5/2}} + O(v^6)\right], \tag{3.9}$$

$$q_0 = -1 + \frac{M}{r_0} - \frac{2\,q\,M^{3/2}}{{r_0}^{3/2}} + \frac{(6+q^2)\,M^2}{{r_0}^2} - \frac{20\,q\,M^{5/2}}{{r_0}^{5/2}} + O(v^6), \tag{3.10}$$

$$q_1 = \frac{2\,M}{r_0}\left[1 + \frac{2\,M}{r_0} - \frac{3\,q\,M^{\frac{3}{2}}}{{r_0}^{\frac{3}{2}}} + \frac{4\,M^2}{{r_0}^2} - \frac{6\,q\,M^{\frac{5}{2}}}{{r_0}^{\frac{5}{2}}} + O(v^6)\right], \tag{3.11}$$

$$q_2 = 1 - \frac{3\,M}{r_0} + \frac{2\,q\,M^{\frac{3}{2}}}{{r_0}^{\frac{3}{2}}} + \frac{(-10-q^2)\,M^2}{{r_0}^2} + \frac{26\,q\,M^{\frac{5}{2}}}{{r_0}^{\frac{5}{2}}} + O(v^6). \tag{3.12}$$

We obtain $r^{(1)}(t)$ from Eq.(3.7) as

$$r^{(1)}(t) = \cos\Omega_r t, \tag{3.13}$$

where we set $r(t=0) = r_0(1+e)$. Substitution of Eq. (3.13) into Eq. (3.8) yields after integration

$$r^{(2)}(t) = q_3(1-\cos\Omega_r t) + q_4(1-\cos 2\Omega_r t), \tag{3.14}$$

where $q_3 = -q_0$ and $q_4 = q_2/2$.

In the same way, we can calculate $\varphi(t)$. From Eq.(3.1), we have $d\varphi/dt = \Phi/T$, which can be expanded in terms of e using Eq.(3.3), (3.4), (3.6), (3.13) and (3.14). Integrating the resulting equation, we obtain

$$\varphi(t) = \Omega_\varphi t + e p_1 \sin\Omega_r t + e^2 p_2 \sin\Omega_r t + e^2 p_3 \sin 2\Omega_r t + O(e^3), \tag{3.15}$$

where

$$\begin{aligned}
p_1 &= -2 - \frac{4\,M}{r_0} + \frac{6\,q\,M^{\frac{3}{2}}}{{r_0}^{\frac{3}{2}}} + \frac{(-17-q^2)\,M^2}{{r_0}^2} + \frac{48\,q\,M^{\frac{5}{2}}}{{r_0}^{\frac{5}{2}}} + O(v^6),\\
p_2 &= 2 + \frac{2\,M}{r_0} - \frac{2\,q\,M^{\frac{3}{2}}}{{r_0}^{\frac{3}{2}}} + \frac{(1-q^2)\,M^2}{{r_0}^2} + \frac{6\,q\,M^{\frac{5}{2}}}{{r_0}^{\frac{5}{2}}} + O(v^6),\\
p_3 &= \frac{5}{4} + \frac{M}{4\,r_0} - \frac{2\,q\,M^{\frac{3}{2}}}{{r_0}^{\frac{3}{2}}} - \frac{(9+7\,q^2)\,M^2}{8\,{r_0}^2} + \frac{59\,(-1+q^2)\,M^3}{8\,{r_0}^3} + O(v^6),
\end{aligned} \tag{3.16}$$

and

$$\begin{aligned}
\Omega_\varphi = \frac{\sqrt{M}}{{r_0}^{\frac{3}{2}}}\Bigg[&1 - \frac{q\,M^{3/2}}{{r_0}^{3/2}}\\
&+e^2\left(-\frac{3}{2} + \frac{9\,M}{2r_0} - \frac{9\,q\,M^{3/2}}{2\,{r_0}^{3/2}} + \frac{3\,(6+q^2)\,M^2}{{r_0}^2} - \frac{60\,q\,M^{5/2}}{{r_0}^{5/2}}\right) + O(v^6)\Bigg]
\end{aligned} \tag{3.17}$$

The fact that $\Omega_r \neq \Omega_\varphi$ implies that these eccentric orbits are not closed.

3.2 Integration of the source term

In this section, using the results of the previous section, we derive the source term of the Teukolsky equation and integrate it to give the amplitude of the Teukolsky function at infinity.

The energy momentum tensor of a test particle of mass μ is given by

$$T^{\mu\nu} = \frac{\mu}{\Sigma \sin\theta dt/d\tau} \frac{dz^\mu}{d\tau}\frac{dz^\nu}{d\tau}\delta(r-r(t))\delta(\theta-\pi/2)\delta(\varphi-\varphi(t)). \tag{3.18}$$

The source term of the Teukolsky equation is given by

$$T_{\ell m\omega} = 4\int d\Omega dt \rho^{-5}\overline{\rho}^{-1}(B_2' + B_2'^*) e^{-im\varphi+i\omega t}\frac{{}_{-2}S^{a\omega}_{\ell m}}{\sqrt{2\pi}}, \tag{3.19}$$

where

$$\begin{aligned} B_2' &= -\frac{1}{2}\rho^8\overline{\rho}L_{-1}[\rho^{-4}L_0(\rho^{-2}\overline{\rho}^{-1}T_{nn})] \\ &\quad - \frac{1}{2\sqrt{2}}\rho^8\overline{\rho}\Delta^2 L_{-1}[\rho^{-4}\overline{\rho}^2 J_+(\rho^{-2}\overline{\rho}^{-2}\Delta^{-1}T_{\overline{m}n})], \\ B_2'^* &= -\frac{1}{4}\rho^8\overline{\rho}\Delta^2 J_+[\rho^{-4}J_+(\rho^{-2}\overline{\rho}T_{\overline{m}\overline{m}})] \\ &\quad - \frac{1}{2\sqrt{2}}\rho^8\overline{\rho}\Delta^2 J_+[\rho^{-4}\overline{\rho}^2\Delta^{-1}L_{-1}(\rho^{-2}\overline{\rho}^{-2}T_{\overline{m}n})], \end{aligned} \tag{3.20}$$

with

$$\begin{aligned} \rho &= (r - ia\cos\theta)^{-1}, \\ L_s &= \partial_\theta + \frac{m}{\sin\theta} - a\omega\sin\theta + s\cot\theta, \\ J_+ &= \partial_r + iK/\Delta, \end{aligned} \tag{3.21}$$

and $\overline{\rho}$ etc. denoting the complex conjugate of ρ.

In the present case, the tetrad components of the energy momentum tensor, T_{nn}, $T_{\overline{m}n}$ and $T_{\overline{m}\overline{m}}$, are in the forms,

$$\begin{aligned} T_{nn} &= \frac{C_{nn}}{\sin\theta}\delta(r-r(t))\delta(\theta-\pi/2)\delta(\varphi-\varphi(t)), \\ T_{\overline{m}n} &= \frac{C_{\overline{m}n}}{\sin\theta}\delta(r-r(t))\delta(\theta-\pi/2)\delta(\varphi-\varphi(t)), \\ T_{\overline{m}\overline{m}} &= \frac{C_{\overline{m}\overline{m}}}{\sin\theta}\delta(r-r(t))\delta(\theta-\pi/2)\delta(\varphi-\varphi(t)), \end{aligned} \tag{3.22}$$

where

$$C_{nn} = \frac{\mu}{4\Sigma^3\dot{t}}\left[E(r^2+a^2) - al_z + \Sigma\frac{dr}{d\tau}\right]^2,$$

$$C_{\overline{m}\,n} = -\frac{\mu\rho}{2\sqrt{2}\Sigma^2\dot{t}}\left[E(r^2+a^2) - al_z + \Sigma\frac{dr}{d\tau}\right]\left[i\sin\theta\Big(aE - \frac{l_z}{\sin^2\theta}\Big)\right], \tag{3.23}$$
$$C_{\overline{m}\,\overline{m}} = \frac{\mu\rho^2}{2\Sigma\dot{t}}\left[i\sin\theta\Big(aE - \frac{l_z}{\sin^2\theta}\Big)\right]^2,$$

and $\dot{t} = dt/d\tau$.

Substituting Eq.(3.20) into Eq.(3.19) and performing integration by part, we obtain

$$\begin{aligned} T_{\ell m\omega} &= \frac{4}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dt \int d\theta e^{i\omega t - im\varphi(t)} \\ &\times \Big[-\frac{1}{2}L_1^\dagger\{\rho^{-4}L_2^\dagger(\rho^3 S)\}C_{n\,n}\rho^{-2}\overline{\rho}^{-1}\delta(r-r(t))\delta(\theta-\pi/2) \\ &+ \frac{\Delta^2\overline{\rho}^2}{\sqrt{2}\rho}(L_2^\dagger S + ia(\overline{\rho}-\rho)\sin\theta S)J_+\{C_{\overline{m}\,n}\rho^{-2}\overline{\rho}^{-2}\Delta^{-1}\delta(r-r(t))\delta(\theta-\pi/2)\} \\ &+ \frac{1}{2\sqrt{2}}L_2^\dagger\{\rho^3 S(\overline{\rho}^2\rho^{-4})_{,r}\}C_{\overline{m}\,n}\Delta\rho^{-2}\overline{\rho}^{-2}\delta(r-r(t))\delta(\theta-\pi/2) \\ &- \frac{1}{4}\rho^3\Delta^2 S J_+\{\rho^{-4}J_+(\overline{\rho}\rho^{-2}C_{\overline{m}\,\overline{m}}\delta(r-r(t))\delta(\theta-\pi/2))\}\Big], \end{aligned} \tag{3.24}$$

where

$$L_s^\dagger = \partial_\theta - \frac{m}{\sin\theta} + a\omega\sin\theta + s\cot\theta, \tag{3.25}$$

and S denotes ${}_{-2}S_{\ell m}^{a\omega}(\theta(t))$ for simplicity.

We further rewrite Eq.(3.24) as

$$\begin{aligned} T_{\ell m\omega} &= \int_{-\infty}^{\infty} dt e^{i\omega t - im\varphi(t)}\Delta^2\Big[(A_{n\,n\,0} + A_{\overline{m}\,n\,0} + A_{\overline{m}\,\overline{m}\,0})\delta(r-r(t)) \\ &+ \{(A_{\overline{m}\,n\,1} + A_{\overline{m}\,\overline{m}\,1})\delta(r-r(t))\}_{,r} + \{A_{\overline{m}\,\overline{m}\,2}\delta(r-r(t))\}_{,rr}\Big], \end{aligned} \tag{3.26}$$

where $A_{n\,n\,0}$ etc. are given in Appendix B. Inserting Eq.(3.26) into Eq.(2.5), we obtain $\tilde{Z}_{\ell m\omega}$ as

$$\begin{aligned} \tilde{Z}_{\ell m\omega} &= \frac{1}{2i\omega B_{\ell m\omega}^{in}}\int_{-\infty}^{\infty} dt e^{i\omega t - im\varphi(t)}\Big[R_{\ell m\omega}^{in}\{A_{n\,n\,0} + A_{\overline{m}\,n\,0} + A_{\overline{m}\,\overline{m}\,0}\} \\ &\quad -\frac{dR_{\ell m\omega}^{in}}{dr}\{A_{\overline{m}\,n\,1} + A_{\overline{m}\,\overline{m}\,1}\} + \frac{d^2R_{\ell m\omega}^{in}}{dr^2}A_{\overline{m}\,\overline{m}\,2}\Big]_{r=r(t)}, \\ &\equiv \frac{1}{2i\omega_n B_{\ell m\omega}^{in}}\int_{-\infty}^{\infty} dt e^{i\omega t - im\varphi(t)} W_{\ell m\omega}. \end{aligned} \tag{3.27}$$

Eq. (3.27) can be simplified by noting the result of the previous section that the orbits of our interest have the properties,

$$r(t+\Delta t) = r(t), \qquad d\varphi/dt\,|_{t=t+\Delta t} = d\varphi/dt\,|_{t=t}, \tag{3.28}$$

where $\Delta t = 2\pi/\Omega_r$. Using this fact, we can rewrite the above integral as

$$
\begin{aligned}
\tilde{Z}_{\ell m\omega} &= \frac{1}{2i\omega B^{in}_{\ell m\omega}} \int_{-\infty}^{\infty} dt e^{i\omega t - im\varphi(t)} W_{\ell m\omega} \\
&= \frac{1}{2i\omega B^{in}_{\ell m\omega}} \frac{2\pi}{\Delta t} \int_{0}^{\Delta t} dt e^{i\omega t - im\varphi(t)} W_{\ell m\omega} \sum_n \delta(\omega - \omega_n) \\
&\equiv \sum_n \delta(\omega - \omega_n) Z_{\ell m\omega},
\end{aligned}
\tag{3.29}
$$

where

$$
\omega_n = n\Omega_r + m\Omega_\varphi, \quad (n = 0, \pm 1, \pm 2, \ldots). \tag{3.30}
$$

Using the solution of the geodesic equation for $r(t)$, we expand $W_{\ell m\omega_n}$ in terms of e. We can see that the resulting equation has the structure up to the order $O(e^2)$,

$$
\begin{aligned}
W_{\ell m\omega} &= f_0 + e\left(f_1 r^{(1)} + f_2 \frac{dr^{(1)}}{dt}\right) + e^2 \left(f_3 r^{(2)} + f_4 \frac{dr^{(2)}}{dt} + f_5 (r^{(1)})^2\right. \\
&+ \left. f_6 \left(\frac{dr^{(1)}}{dt}\right)^2 + f_7 r^{(1)} \frac{dr^{(1)}}{dt} + f_8 \right) + O(e^3).
\end{aligned}
\tag{3.31}
$$

We insert Eq.(3.13), (3.14), (3.15) and (3.31) into Eq.(3.32) and expand it in terms of e. Finally, using the relations $\cos(\Omega_r t) = (e^{i\Omega_r t} + e^{-i\Omega_r t})/2$ and $\sin(\Omega_r t) = (e^{i\Omega_r t} - e^{-i\Omega_r t})/2i$, we obtain

$$
\begin{aligned}
I_{\ell m\omega_n} &\equiv \int_0^{\Delta t} dt e^{i\omega_n t - im\varphi(t)} W_{\ell m\omega_n} \\
&= \Delta t \left[\left\{ f_0 + e^2 \left(\frac{f_5}{2} + f_8 - \frac{m^2 p_1^2}{4} f_0 \right.\right.\right. \\
&\quad \left.\left. + (q_3 + q_4) f_3 + \frac{im\Omega_r p_1}{2} f_2 + \frac{\Omega_r^2}{2} f_6 \right)\right] \delta_{n,0} \\
&\quad + e \left(\frac{f_1}{2} + \frac{mp_1}{2} f_0 - \frac{i\Omega_r}{2} f_2 \right) \delta_{n,1} \\
&\quad + e \left(\frac{f_1}{2} - \frac{mp_1}{2} f_0 + \frac{i\Omega_r}{2} f_2 \right) \delta_{n,-1} \\
&\quad + e^2 \left(\frac{3 f_5}{4} + f_8 + \frac{f_1\, m\, p_1}{4} - \frac{f_0\, m^2\, {p_1}^2}{8} + \frac{f_0\, m\, p_3}{2} + f_3\, q_3 \right. \\
&\quad \left. + \frac{f_3\, q_4}{2} - \frac{i}{4} f_7\, w + \frac{i}{4} f_2\, m\, p_1\, w + i\, f_4\, q_4\, w + \frac{f_6\, w^2}{4} \right) \delta_{n,2} \\
&\quad + e^2 \left(\frac{3 f_5}{4} + f_8 - \frac{f_1\, m\, p_1}{4} - \frac{f_0\, m^2\, {p_1}^2}{8} - \frac{f_0\, m\, p_3}{2} + f_3\, q_3 \right. \\
&\quad \left. + \frac{f_3\, q_4}{2} + \frac{i}{4} f_7\, w + \frac{i}{4} f_2\, m\, p_1\, w - i\, f_4\, q_4\, w + \frac{f_6\, w^2}{4} \right) \delta_{n,-2} \Big]
\end{aligned}
\tag{3.32}
$$

We can see from Eq.(3.32) that $n = \pm 2$ modes occur at $O(e^2)$. Then when we evaluate the luminosity, $n = \pm 2$ modes contribute from $O(e^4)$. Therefore we have only to calculate $n = 0, \pm 1$ modes to evaluate the luminosity up to $O(e^2)$.

4 The energy and angular momentum fluxes

In this section, we calculate the energy and angular momentum fluxes to $O(v^5)$ beyond the quadrupole formula and to $O(e^2)$ in the eccentricity. From Eq.(2.1), ψ_4 at $r \to \infty$ takes the form,

$$\psi_4 = \frac{1}{r} \sum_{n=-2}^{2} \sum_{\ell=2}^{4} \sum_{m=-\ell}^{\ell} Z_{\ell m \omega_n} \frac{{}_{-2}S^{a\omega_n}_{\ell m}}{\sqrt{2\pi}} e^{i\omega_n (r^* - t) + im\varphi}. \tag{4.1}$$

At infinity, ψ_4 is related to the two independent modes of gravitational waves h_+ and $h_\times$ as

$$\psi_4 = \frac{1}{2}(\ddot{h}_+ - i\ddot{h}_\times). \tag{4.2}$$

From Eqs.(4.1) and (4.2), the energy flux averaged over $t \gg \Delta t$ is given by

$$\left\langle \frac{dE}{dt} \right\rangle = \sum_{\ell,m,n} \frac{\left| Z_{\ell m \omega_n} \right|^2}{4\pi\omega_n^2} \equiv \sum_{\ell,m,n} \left(\frac{dE}{dt} \right)_{\ell m n}. \tag{4.3}$$

In the same way, the angular momentum flux is given by

$$\left\langle \frac{dJ_z}{dt} \right\rangle = \sum_{\ell,m,n} \frac{m\left| Z_{\ell m \omega_n} \right|^2}{4\pi\omega_n^3} \equiv \sum_{\ell,m,n} \left(\frac{dJ_z}{dt} \right)_{\ell m n} = \sum_{\ell,m,n} \frac{m}{\omega_n} \left(\frac{dE}{dt} \right)_{\ell m n}. \tag{4.4}$$

In order to express the post-Newtonian corrections to the luminosity, we define $\eta_{\ell m n}$ as

$$\left(\frac{dE}{dt} \right)_{\ell m n} \equiv \frac{1}{2} \left(\frac{dE}{dt} \right)_N \eta_{\ell,m,n}, \tag{4.5}$$

where $(dE/dt)_N$ is the Newtonian quadrupole luminosity:

$$\left(\frac{dE}{dt} \right)_N = \frac{32\mu^2 M^3}{5 r_0^5} = \frac{32}{5} \left(\frac{\mu}{M} \right)^2 v^{10}.$$

For $\ell = 2$, $\eta_{\ell,m,n}$ which contribute to the luminosity to $O(v^5)$ are given by the following formulae. We only show $\eta_{\ell m n}$ for $m \geq 0$ mode. $\eta_{\ell,m,n}$ for $m \leq 0$ is easily obtained from the symmetry $\eta_{\ell,m,n} = \eta_{\ell,-m,-n}$.

$$\begin{aligned}
\eta_{2,2,0} &= 1 - \frac{107\, v^2}{21} + 4\,\pi\, v^3 - 6\, q\, v^3 + \frac{4784\, v^4}{1323} + 2\, q^2\, v^4 \\
&\quad - \frac{428\,\pi\, v^5}{21} + \frac{4216\, q\, v^5}{189} \\
&\quad + e^2 \left(-10 + \frac{932\, v^2}{21} - 46\,\pi\, v^3 + 84\, q\, v^3 - \frac{14417\, v^4}{147} - 23\, q^2\, v^4 \right.
\end{aligned}$$

$$
\begin{aligned}
& \quad + \frac{4748\,\pi\,v^5}{21} + \frac{2675\,q\,v^5}{189}\Bigg) \\
\eta_{2,2,1} &= e^2\left(\frac{729}{64} - \frac{3645\,v^2}{64} + \frac{2187\,\pi\,v^3}{32} - \frac{3645\,q\,v^3}{32}\right. \\
& \quad \left. + \frac{24057\,v^4}{256} + \frac{2187\,q^2\,v^4}{64} - \frac{6561\,\pi\,v^5}{16} + \frac{9477\,q\,v^5}{112}\right) \\
\eta_{2,2,-1} &= e^2\left(\frac{9}{64} + \frac{1041\,v^2}{448} + \frac{9\,\pi\,v^3}{32} - \frac{153\,q\,v^3}{32}\right. \\
& \quad \left. + \frac{2224681\,v^4}{112896} + \frac{99\,q^2\,v^4}{64} + \frac{615\,\pi\,v^5}{112} - \frac{27857\,q\,v^5}{336}\right) \\
\eta_{2,1,0} &= \frac{v^2}{36} - \frac{q\,v^3}{12} - \frac{17\,v^4}{504} + \frac{q^2\,v^4}{16} + \frac{\pi\,v^5}{18} - \frac{793\,q\,v^5}{9072} \\
& \quad + e^2\left(\frac{-2\,v^2}{9} + \frac{2\,q\,v^3}{3} + \frac{93\,v^4}{112} - \frac{q^2\,v^4}{2} - \frac{19\,\pi\,v^5}{36} - \frac{27113\,q\,v^5}{18144}\right) \\
\eta_{2,1,1} &= e^2\left(\frac{4\,v^2}{9} - \frac{4\,q\,v^3}{3} - \frac{172\,v^4}{63} + q^2\,v^4 + \frac{16\,\pi\,v^5}{9} + \frac{2794\,q\,v^5}{567}\right) \\
\eta_{2,0,\pm 1} &= e^2\left(\frac{1}{96} - \frac{145\,v^2}{672} + \frac{\pi\,v^3}{48} + \frac{3\,q\,v^3}{16} + \frac{282521\,v^4}{169344} - \frac{3\,q^2\,v^4}{32}\right. \\
& \quad \left. - \frac{83\,\pi\,v^5}{168} - \frac{1255\,q\,v^5}{504}\right),
\end{aligned}
$$

and $\eta_{2,1,-1}$ becomes $O(v^6)$. Putting together the above results, we obtain $(dE/dt)_\ell \equiv \sum_{mn}(dE/dt)_{\ell mn}$ for $\ell = 2$ as

$$
\begin{aligned}
\left(\frac{dE}{dt}\right)_2 &= \left(\frac{dE}{dt}\right)_N \left\{1 - \frac{1277\,v^2}{252} + 4\,\pi\,v^3 - \frac{73\,q\,v^3}{12} + \frac{37915\,v^4}{10584}\right. \\
& \quad + \frac{33\,q^2\,v^4}{16} - \frac{2561\,\pi\,v^5}{126} + \frac{201575\,q\,v^5}{9072} \\
& \quad + e^2\left(\frac{37}{24} - \frac{2581\,v^2}{252} + \frac{1087\,\pi\,v^3}{48} - \frac{211\,q\,v^3}{6} + \frac{325393\,v^4}{21168} + \frac{105\,q^2\,v^4}{8}\right. \\
& \quad \left.\left. - \frac{29857\,\pi\,v^5}{168} + \frac{11293\,q\,v^5}{672}\right)\right\}.
\end{aligned} \tag{4.6}
$$

For $\ell = 3$, we obtain

$$
\begin{aligned}
\eta_{3,3,0} &= \frac{1215\,v^2}{896} - \frac{1215\,v^4}{112} + \frac{3645\,\pi\,v^5}{448} - \frac{1215\,q\,v^5}{112} \\
& \quad + e^2\left(\frac{-10935\,v^2}{448} + \frac{37665\,v^4}{256} - \frac{142155\,\pi\,v^5}{896} + \frac{134865\,q\,v^5}{448}\right) \\
\eta_{3,3,1} &= e^2\left(\frac{640\,v^2}{21} - \frac{46720\,v^4}{189} + \frac{5120\,\pi\,v^5}{21} - \frac{1280\,q\,v^5}{3}\right)
\end{aligned}
$$

$$
\begin{aligned}
\eta_{3,3,-1} &= e^2\left(\frac{15\,v^2}{14}+\frac{1055\,v^4}{126}+\frac{30\,\pi\,v^5}{7}-\frac{435\,q\,v^5}{14}\right)\\
\eta_{3,2,0} &= \frac{5\,v^4}{63}-\frac{40\,q\,v^5}{189}+e^2\left(\frac{-65\,v^4}{63}+\frac{520\,q\,v^5}{189}\right)\\
\eta_{3,2,1} &= e^2\left(\frac{3645\,v^4}{1792}-\frac{1215\,q\,v^5}{224}\right)\\
\eta_{3,2,-1} &= e^2\left(\frac{5\,v^4}{1792}-\frac{5\,q\,v^5}{672}\right)\\
\eta_{3,1,0} &= \frac{v^2}{8064}-\frac{v^4}{1512}+\frac{\pi\,v^5}{4032}-\frac{17\,q\,v^5}{9072}\\
&\quad+e^2\left(\frac{-v^2}{4032}+\frac{65\,v^4}{16128}-\frac{\pi\,v^5}{1152}+\frac{199\,q\,v^5}{36288}\right)\\
\eta_{3,1,1} &= e^2\left(\frac{v^2}{126}-\frac{23\,v^4}{126}+\frac{2\,\pi\,v^5}{63}+\frac{122\,q\,v^5}{567}\right)\\
\eta_{3,0,\pm1} &= e^2\left(\frac{v^4}{2688}-\frac{q\,v^5}{1008}\right)
\end{aligned}
$$

and $\eta_{3,1,-1}$ becomes $O(v^6)$. Thus we obtain

$$
\begin{aligned}
\left(\frac{dE}{dt}\right)_3 &= \left(\frac{dE}{dt}\right)_N\left\{\frac{1367\,v^2}{1008}-\frac{32567\,v^4}{3024}+\left(\frac{16403\,\pi}{2016}-\frac{896\,q}{81}\right)v^5\right.\\
&\quad\left.+e^2\left(\frac{1801\,v^2}{252}-\frac{78509\,v^4}{864}+\left(\frac{40083\,\pi}{448}-\frac{8913\,q}{56}\right)v^5\right)\right\}
\end{aligned}
\tag{4.7}
$$

For $\ell=4$, we have

$$
\begin{aligned}
\eta_{4,4,0} &= \frac{1280\,v^4}{567}-\frac{37120\,e^2\,v^4}{567},\\
\eta_{4,4,1} &= \frac{48828125\,e^2\,v^4}{580608},\\
\eta_{4,4,-1} &= \frac{32805\,e^2\,v^4}{7168},\\
\eta_{4,2,0} &= \frac{5\,v^4}{3969}-\frac{25\,e^2\,v^4}{3969},\\
\eta_{4,2,-1} &= \frac{5\,e^2\,v^4}{254016},\\
\eta_{4,0,\pm1} &= \frac{e^2\,v^4}{225792},
\end{aligned}
$$

and $\eta_{4,2,1}$ becomes $O(v^6)$. Hence we have

$$
\left(\frac{dE}{dt}\right)_4=\left(\frac{dE}{dt}\right)_N\left\{\frac{8965\,v^4}{3969}+\frac{2946739\,e^2\,v^4}{127008}\right\}. \tag{4.8}
$$

Finally, gathering all the above results, we have the energy luminosity up to $O(v^5)$ as

$$
\begin{aligned}
\left\langle \frac{dE}{dt} \right\rangle &= \left(\frac{dE}{dt}\right)_N \left\{ 1 - \frac{1247\,v^2}{336} + 4\,\pi\,v^3 - \frac{73\,q\,v^3}{12} - \frac{44711\,v^4}{9072} + \frac{33\,q^2\,v^4}{16} \right. \\
&\quad - \frac{8191\,\pi\,v^5}{672} + \frac{3749\,q\,v^5}{336} + e^2 \left(\frac{37}{24} - \frac{65\,v^2}{21} + \frac{1087\,\pi\,v^3}{48} - \frac{211\,q\,v^3}{6} \right. \\
&\quad \left.\left. - \frac{474409\,v^4}{9072} + \frac{105\,q^2\,v^4}{8} - \frac{118607\,\pi\,v^5}{1344} - \frac{95663\,q\,v^5}{672} \right) \right\}
\end{aligned} \tag{4.9}
$$

This result agrees with those derived in Shibata et.al. [17] if we set $e = 0$. To compare our results to those derived in the previous papers, it is convenient to change the parameter from v to $v' \equiv (M\Omega_\varphi)^{1/3}$. The relation between v and v' is given by

$$
\begin{aligned}
v &= v' \left[1 + \frac{q}{3} v'^3 + e^2 \left\{ \frac{1}{2} - \frac{3}{2} v'^2 + \frac{8}{3} q v'^3 \right. \right. \\
&\quad \left.\left. - 6 v'^4 - q^2 v'^4 + \frac{31}{2} q v'^5 \right\} \right] .
\end{aligned} \tag{4.10}
$$

Then we can see that the terms which are proportional to e^2 agree with the formulae derived by Peters and Mathews [27] at reading order, Galt'sov et.al. [28] and Blanchet and Schäfer at v^2 order [29], Blanchet and Schäfer at v^3 order without q term [10] and Shibata at v^3 order with q term [18] when we expand their formulae by e assuming $e \ll 1$ and set $\mu/M \ll 1$.

From Eq.(4.4), the averaged angular momentum fluxes for $\ell = 2, 3$ and 4 are calculated to give

$$
\begin{aligned}
\left(\frac{dJ_z}{dt}\right)_2 &= \left(\frac{dJ_z}{dt}\right)_N \left\{ 1 - \frac{1277\,v^2}{252} + 4\,\pi\,v^3 - \frac{61\,q\,v^3}{12} + \frac{37915\,v^4}{10584} + \frac{33\,q^2\,v^4}{16} \right. \\
&\quad - \frac{2561\,\pi\,v^5}{126} + \frac{22229\,q\,v^5}{1296} + e^2 \left(-\frac{5}{8} + \frac{137\,v^2}{24} + \frac{49\,\pi\,v^3}{8} - \frac{57\,q\,v^3}{4} \right. \\
&\quad \left.\left. - \frac{249787\,v^4}{14112} + \frac{203\,q^2\,v^4}{32} - \frac{20437\,\pi\,v^5}{504} - \frac{164449\,q\,v^5}{4536} \right) \right\} \\
\left(\frac{dJ_z}{dt}\right)_3 &= \left(\frac{dJ_z}{dt}\right)_N \left\{ \frac{1367\,v^2}{1008} - \frac{32567\,v^4}{3024} + \left(\frac{16403\,\pi}{2016} - \frac{88049\,q}{9072} \right) v^5 \right. \\
&\quad \left. + e^2 \left(\frac{67\,v^2}{32} - \frac{66497\,v^4}{2016} + \left(\frac{43193\,\pi}{1008} - \frac{1675571\,q}{18144} \right) , v^5 \right) \right. \\
\left(\frac{dJ_z}{dt}\right)_4 &= \left(\frac{dJ_z}{dt}\right)_N \left\{ \frac{8965\,v^4}{3969} + \frac{478195\,e^2\,v^4}{42336} \right\}
\end{aligned}
$$

where $(dJ_z/dt)_N$ is defined to be

$$
\left(\frac{dJ_z}{dt}\right)_N = \frac{32\mu^2 M^{5/2}}{5 r_0^{7/2}} = \frac{32}{5} \left(\frac{\mu}{M}\right)^2 M v^7 . \tag{4.11}
$$

Total angular momentum luminosity is then given by

$$
\begin{aligned}
\left\langle \frac{dJ_z}{dt} \right\rangle &= \left(\frac{dJ_z}{dt} \right)_N \left\{ 1 - \frac{1247\, v^2}{336} + \left(4\,\pi - \frac{61\, q}{12} \right) v^3 + \left(-\frac{44711}{9072} + \frac{33\, q^2}{16} \right) v^4 \right. \\
&+ \left(\frac{-8191\,\pi}{672} + \frac{417\, q}{56} \right) v^5 + e^2 \left(-\frac{5}{8} + \frac{749\, v^2}{96} + \left(\frac{49\,\pi}{8} - \frac{57\, q}{4} \right) v^3 \right. \\
&+ \left. \left. \left(-\frac{238229}{6048} + \frac{203\, q^2}{32} \right) v^4 + \left(\frac{773\,\pi}{336} - \frac{28807\, q}{224} \right) v^5 \right) \right\} \qquad (4.12)
\end{aligned}
$$

From above results, we find that q^2 terms appear at v^4 order and linear terms in q appear at v^3 and v^5 orders in all of the above formulae. Those feature are the same as in Shibata et.at.

5 Stability of circular orbit under radiation reaction

Apostolatos et.al. [26] investigated the stability of circular orbit of a test particle around a Schwarzschild black hole under the influence of radiation reaction, and they show that a circular orbit is stable only if the orbital radius is greater than a critical radius $r_c \simeq 6.6792M$. In this section, we extend their analysis to a Kerr black hole case and investigate the stability of circular orbit under radiation reaction using the post-Newtonian approximation.

We are interested in the evolution of the radius r_0 and eccentricity e. Since E and l_z of a particle which we are considering is determined by e and r_0, the knowledge of the rates of loss of energy and angular momentum due to gravitational radiation is sufficient to determine the evolution of e and r_0. The evolution equations for e and r_0 are determined by

$$
\begin{aligned}
\dot{E} &= \frac{\partial E(r_0,e)}{\partial r_0}\frac{dr_0}{dt} + \frac{\partial E(r_0,e)}{\partial e}\frac{de}{dt}, \\
\dot{l_z} &= \frac{\partial l_z(r_0,e)}{\partial r_0}\frac{dr_0}{dt} + \frac{\partial l_z(r_0,e)}{\partial e}\frac{de}{dt}, \qquad (5.1)
\end{aligned}
$$

where $\dot{E}$ and $\dot{l_z}$ represent the time derivatives. We concentrate our attention to the secular evolution of e and r_0. We therefore set $\dot{E} = -\langle dE/dt \rangle$ and $\dot{l_z} = -\langle dJ_z/dt \rangle$. Solving Eq.(5.1) in terms of de/dt and dr_0/dt, we have

$$
\begin{aligned}
\frac{dr_0}{dt} &= \left[-\frac{\partial L_z}{\partial e}\left\langle \frac{dE}{dt} \right\rangle + \frac{\partial E}{\partial e}\left\langle \frac{dJ_z}{dt} \right\rangle \right] \Big/ \left[\frac{\partial L_z}{\partial e}\frac{\partial E}{\partial r_0} - \frac{\partial E}{\partial e}\frac{\partial L_z}{\partial r_0} \right], \\
\frac{de}{dt} &= \left[\frac{\partial L_z}{\partial r_0}\left\langle \frac{dE}{dt} \right\rangle - \frac{\partial E}{\partial r_0}\left\langle \frac{dJ_z}{dt} \right\rangle \right] \Big/ \left[\frac{\partial L_z}{\partial e}\frac{\partial E}{\partial r_0} - \frac{\partial E}{\partial e}\frac{\partial L_z}{\partial r_0} \right], \qquad (5.2)
\end{aligned}
$$

We expand E and l_z in terms of e as

$$
E = E^{(0)} + eE^{(1)} + e^2E^{(2)} + e^3E^{(3)} + O(e^4),
$$

$$l_z = l_z^{(0)} + e l_z^{(1)} + e^2 l_z^{(2)} + e^3 l_z^{(3)} + O(e^4). \tag{5.3}$$

Note that $E^{(1)}$ and $l_z^{(1)}$ are zero. We also rewrite $\langle dE/dt \rangle$ and $\langle dJ_z/dt \rangle$ of the previous section as

$$\left\langle \frac{dE}{dt} \right\rangle = \dot{E}^{(0)} + e\dot{E}^{(1)} + e^2 \dot{E}^{(2)} + O(e^3), \tag{5.4}$$

$$\left\langle \frac{dJ_z}{dt} \right\rangle = \dot{J_z}^{(0)} + e\dot{J_z}^{(1)} + e^2 \dot{J_z}^{(2)} + O(e^3). \tag{5.5}$$

Then using the fact that

$$\dot{E}^{(0)} = \Omega_\varphi \dot{J_z}^{(0)}, \quad \dot{E}^{(1)} = \dot{J_z}^{(1)} = 0, \tag{5.6}$$

we have $\dot{e}$ and $\dot{r}_0$ to the reading order of e as

$$\dot{e} = e \frac{\left(\dot{E}^{(2)} L_z^{(0)\prime} + \dot{E}^{(0)} L_z^{(2)\prime} + \dot{J}_z^{(2)} E^{(0)\prime} + \dot{J}_z^{(0)} E^{(2)\prime} \right)}{-2L_z^{(0)\prime} E^{(2)} + 2E^{(0)\prime} L_z^{(2)}}, \tag{5.7}$$

$$\dot{r}_0 = \frac{2\dot{J}_z^{(0)} E^{(2)} - 2\dot{E}^{(0)} L_z^{(2)}}{-2L_z^{(0)\prime} E^{(2)} + 2E^{(0)\prime} L_z^{(2)}}. \tag{5.8}$$

where $' = \partial/\partial r_0$. Note that $\dot{e}$ is proportional to e. This means that circular orbit remains circular under radiation reaction since $\dot{e}(e=0) = 0$. This fact hold if Eq.(5.6) is correct. In this paper, we only give the proof of Eq.(5.6) using the post-Newtonian approximation, but the formulation of the previous section, especially Eq.(3.31) and (3.32), suggests that Eq.(5.6) will also hold in the relativistic cases.

If we expand Eq.(5.7) in terms of v, we have

$$\begin{aligned} \frac{de}{dt} &= -\frac{304}{15} \frac{\mu}{M^2} e v^8 \left[1 - \frac{6849\, v^2}{2128} + \frac{985\, \pi\, v^3}{152} - \frac{879\, q\, v^3}{76} - \frac{286397\, v^4}{38304} + \frac{1345\, q^2\, v^4}{304} \right. \\ &\quad \left. - \frac{87947\, \pi\, v^5}{4256} - \frac{23201\, q\, v^5}{2128} \right] \end{aligned} \tag{5.9}$$

$$\begin{aligned} \frac{dr_0}{dt} &= -\frac{64}{5} \frac{\mu}{M} v^6 \left[1 - \frac{743\, v^2}{336} + 4\, \pi\, v^3 - \frac{133\, q\, v^3}{12} + \frac{34103\, v^4}{18144} + \frac{81\, q^2\, v^4}{16} \right. \\ &\quad \left. - \frac{4159\, \pi\, v^5}{672} - \frac{1451\, q\, v^5}{56} \right]. \end{aligned} \tag{5.10}$$

From Eq.(5.9) and (5.10), we obtain

$$\begin{aligned} \frac{de}{dr_0} &= \frac{19}{12} \frac{e}{r_0} \left[1 - \frac{3215\, v^2}{3192} + \frac{377\, \pi\, v^3}{152} - \frac{55\, q\, v^3}{114} - \frac{111813545\, v^4}{9652608} - \frac{97\, q^2\, v^4}{152} \right. \\ &\quad \left. - \frac{253409\, \pi\, v^5}{51072} + \frac{53203\, q\, v^5}{19152} \right] \end{aligned} \tag{5.11}$$

We see from Eq.(5.9) that in the slow motion limit $v \ll 1$, $\dot{e}$ is always negative and this means that circular orbit is stable under the influence of radiation reaction. We also find that if $q > 0\,(q < 0)$, the rate of change of the eccentricity and radius decrease (increase) at order $O(v^3)$ and $O(v^5)$ and always increase at order $O(v^4)$. Note however that since the Teukolsky equation is constructed to the first order of μ/M, the radiation reaction to the particle is not taken into account. Then we must keep in mind that above results are valid only in the restricted situations when the adiabatic approximation is valid and the mass ratio is large (i.e. $\mu/M \ll 1$).

Using Eq.(5.9), we examine the critical radius where $\dot{e}$ changes sign. We see that $\dot{e}$ become zero at $r_0 \simeq 3.7M$ when $q = 0$. This value contradicts with the fact that there are the last circular stable orbit at $r = 6M$ in the Schwarzschild space time. This value also disagrees with the critical radius $r \simeq 6.6792M$ obtained by Apostolatos et.al. This is because that the post-Newtonian approximation does not give the good approximation around $r = 6M$ [30] even if we calculate the post-Newtonian expansion to order $O(v^5)$.

6 Implications to coalescing compact binaries

In this section, we discuss the effects of post-Newtonian terms in the luminosity to the orbital evolution of inspiraling compact binaries. Although our results are valid only in the test particle limit, we ignore this fact in the following.

The total cycle N of the phase of gravitational waves from an inspiraling binaries from $r = r_i$ $(t = t_i)$ to $r = r_f$ $(t = t_f)$ is given by

$$N \equiv \int_{t_i}^{t_f} f dt = \frac{1}{\pi}\int_{r_f}^{r_i} dr\,\Omega_\varphi \frac{dE/dr}{|dE/dt|}, \tag{6.1}$$

where f is the frequency of the wave.

From Eq.(5.11), we see that the eccentricity e of the orbit decreases according to $e \propto r^{19/12}$ where r is the orbital radius. We estimate the effects of the eccentricity to the total cycle of the gravitational wave in the LIGO's band.

Expanding Ω_φ, dE/dt, and dE/dr with respect to $v = (M/r)^{1/2}$, N is expressed as

$$N = \frac{5}{64\pi}\frac{M}{\mu}\int_{r_f}^{r_i} \frac{dr r^{3/2}}{M^{5/2}} \frac{\sum_{k=0}^{\infty} b_k(e)(M/r)^{k/2} \sum_{k=0}^{\infty} c_k(e)(M/r)^{k/2}}{\sum_{k=0}^{\infty} d_k(e)(M/r)^{k/2}}, \tag{6.2}$$

where the series forms in the denominator and numerator represent the post-Newtonian corrections to the Ω_φ, dE/dt, and dE/dr, that is

$$\begin{aligned}
\sum_{k=0}^{\infty} b_k(e)(M/r)^{k/2} &= \frac{\Omega_\varphi}{(\Omega_\varphi)_N}, \\
\sum_{k=0}^{\infty} c_k(e)(M/r)^{k/2} &= \frac{(dE/dr)}{(dE/dr)_N}, \\
\sum_{k=0}^{\infty} d_k(e)(M/r)^{k/2} &= \frac{(dE/dt)}{(dE/dt)_N},
\end{aligned}$$

and the argument e is given to the coefficients b_k, c_k and d_k to emphasize the e-dependence of the post-Newtonian corrections.

Since the effect of post-Newtonian corrections in the case $e = 0$ has been studied, we examine only the effect due to non-vanishing value of e. For this purpose, we define

$$N^{(n)} = \frac{5}{64\pi}\frac{M}{\mu}\left[\int_{r_f}^{r_i}\frac{dr r^{3/2}}{M^{5/2}}\left\{\frac{\sum_{k=0}^{n} b_k(e)(M/r)^{k/2} + b_{n+1}(0)(M/r)^{(n+1)/2}}{\sum_{k=0}^{n} d_k(e)(M/r)^{k/2} + d_{n+1}(0)(M/r)^{(n+1)/2}}\right.\right.$$
$$\left.\left.\times\left(\sum_{k=0}^{n} c_k(e)(M/r)^{k/2} + c_{n+1}(0)(M/r)^{(n+1)/2}\right)\right\}\right], \tag{6.3}$$

and $\Delta N^{(n)} \equiv \mid N^{(n)} - N^{(n-1)} \mid$ which describe the effect of e at the $O(v^n)$ order. Then we insert $e = e_i(r/r_i)^{19/12}$ into Eq.(6.3), where e_i is eccentricity at $r = r_i$, and integrate it numerically. The results are listed in Table 2. In this calculation, we define r_i as the

$e_i(q=0)$	$\Delta N^{(4)}$	$\Delta N^{(5)}$	$e_i(q=1)$	$\Delta N^{(4)}$	$\Delta N^{(5)}$
0.1	0.36	0.09	0.1	0.37	0.07
0.2	1.37	0.32	0.2	1.38	0.26
0.3	2.77	0.63	0.3	2.81	0.48
0.4	4.28	0.95	0.4	4.37	0.68
0.5	5.63	1.22	0.5	5.83	0.80

Table 2 (a)

$e_i(q=0)$	$\Delta N^{(4)}$	$\Delta N^{(5)}$	$e_i(q=1)$	$\Delta N^{(4)}$	$\Delta N^{(5)}$
0.1	0.54	0.20	0.1	0.55	0.18
0.2	2.01	0.76	0.2	2.07	0.65
0.3	4.06	1.50	0.3	4.22	1.25
0.4	6.25	2.24	0.4	6.57	1.76
0.5	8.22	2.85	0.5	8.76	2.08

Table 2 (b)

$e_i(q=0)$	$\Delta N^{(4)}$	$\Delta N^{(5)}$	$e_i(q=1)$	$\Delta N^{(4)}$	$\Delta N^{(5)}$
0.1	0.19	0.09	0.1	0.21	0.08
0.2	0.72	0.33	0.2	0.76	0.30
0.3	1.46	0.64	0.3	1.56	0.57
0.4	2.23	0.97	0.4	2.42	0.81
0.5	2.94	1.21	0.5	3.22	0.96

Table 2 (c)

Table 1: The effect of the eccentricity to the total cycle of the coalescing binaries in the case $q = 0$ (left) and $q = 1$ (right) and (a) $(m_1, m_2) = (1.4M_\odot, 1.4M_\odot)$, (b) $(m_1, m_2) = (1.4M_\odot, 10M_\odot)$, (c) $(m_1, m_2) = (10M_\odot, 10M_\odot)$.

radius at which $f(r_i) = 10$Hz. Then we set $r_i = 175M$ for $(m_1, m_2) = (1.4M_\odot, 1.4M_\odot)$, $r_i = 68M$ for $(1.4M_\odot, 10M_\odot)$ and $r_i = 47M$ for $(10M_\odot, 10M_\odot)$ respectively. We set

$r_f = 8M$ for all cases since the post-Newtonian expansion of the luminosity does not give a good approximation around $r \sim 6M$ (for instance, dE/dt changes the sign around $r \sim 6M$ for certain value of q and e).

The required accuracy as a template of the gravitational waves from coalescing binaries is $\Delta N < 0.5$. We see from Table 2 that the effects of the eccentricity is important at the order $O(v^4)$ if initial eccentricity is larger than 0.2 or 0.3. On the other hand, the effect of the eccentricity at $O(v^5)$ is important if initial eccentricity is greater than 0.4 or 0.5. These properties hold for all the values of q. Above facts suggest that if the initial eccentricity is $e_i \sim 1$, terms of order $O(v^6)$ may become important and we must calculate those terms to construct the template of binaries with a highly eccentric orbit.

Shapiro and Teukolsky [31] considered a formation scenario of super massive black hole which may exist in the center of galaxies via the collapse of a dense cluster of compact stars. Quinlan and Shapiro [19] investigated the role of binary formation in the evolution of dense stellar systems. In their scenario, formation of binaries through the dissipative two-body encounter dominates formation through three body encounters in the initial stage of cluster evolution. The maximum periastron separation of binaries formed by dissipation of gravitational radiation can be estimated as

$$\begin{aligned} r_{p,max} &= \left(\frac{85\pi}{6}\right)^{2/7} \frac{Gm}{c^{10/7} v^{4/7}} \\ &\sim 115\text{km} \left(\frac{m}{M_\odot}\right) \left(\frac{v}{10^3\,\text{km}}\right)^{-4/7}, \end{aligned}$$

where v is initial relative velocity of the encounter. Those binaries are in the bandwidth of LIGO when they are formed. Binaries formed by two body encounters have typically large eccentricity $e \sim 1$. The advanced detector system of LIGO will have sensitivity to detect the merger of NS-NS binaries within 1 Gpc and BH-BH binaries within 5 Gpc [4]. Then binaries in dense stellar systems may become possible sources of gravitational waves for LIGO/VIRGO. In such case, eccentricity will play an important role to obtain physical informations of the binary systems.

7 Summary

We performed the post-Newtonian expansion of gravitational waves from a test particle in slightly eccentric orbit around a Kerr black hole. The effect of the deviation from circular orbit was expressed by the eccentricity e and we treated e as a perturbation. We calculated the gravitational wave luminosity of both the energy and the angular momentum to order $O(v^5)$ and $O(e^2)$. Our formulas agree with the formulas of the luminosity to the order $O(v^3)$ which were calculated so far by several authors.

We examine the stability of circular orbit in the slow motion situation. We find that if eccentricity e is zero, de/dt is also zero and this means that circular orbit remains circular under the influence of radiation reaction. We also find that circular orbit is stable against radiation reaction even if the central black hole is spinning. Above these results are consistent with the previous works. We also find that if $q > 0$ ($q < 0$), the rate of decrease of eccentricity become small (large) compared to the case of $q = 0$ at $O(v^3)$ and $O(v^5)$ and become always large at $O(v^4)$.

We have also estimated the accuracy of the post-Newtonian expansion to predict the total cycle of coalescing binaries. We found that if the initial eccentricity, at which binaries can be observed with laser interferometers, is greater than 0.2 and 0.4, the effect of the eccentricity become important at order $O(v^4)$ and $O(v^5)$ respectively. There are possibilities that compact binaries with high eccentricity was formed in the dense stellar systems and they may be detected with laser interferometers. Although our results are restricted to case of the large mass ratio and small eccentricity, they will be useful to analyze gravitational waves from such binaries.

On the other hand, gravitational waves from compact stars orbiting around a super-massive black hole in a galactic center will become a possible sources for a proposed laser interferometric detector in space, LISA. Since our formulas give the correct value for such waves, they will give a useful guideline to investigate those cases.

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A The formulae of F and U

In this Appendix we show the potential functions F and U of the SN equation (2.13). Details of the derivation are given in Ref. [25].

The function $F(r)$ is given by

$$F(r) = \frac{\eta_{,r}}{\eta} \frac{\Delta}{r^2 + a^2}, \tag{A.1}$$

where

$$\eta = c_0 + c_1/r + c_2/r^2 + c_3/r^3 + c_4/r^4, \tag{A.2}$$

with

$$\begin{aligned} c_0 &= -12i\omega M + \lambda(\lambda+2) - 12a\omega(a\omega - m), \\ c_1 &= 8ia[3a\omega - \lambda(a\omega - m)], \\ c_2 &= -24iaM(a\omega - m) + 12a^2[1 - 2(a\omega - m)^2], \\ c_3 &= 24ia^3(a\omega - m) - 24Ma^2, \\ c_4 &= 12a^4. \end{aligned} \tag{A.3}$$

The function $U(r)$ is given by

$$U(r) = \frac{\Delta U_1}{(r^2+a^2)^2} + G^2 + \frac{\Delta G_{,r}}{r^2+a^2} - FG, \tag{A.4}$$

where

$$\begin{aligned}
G &= -\frac{2(r-M)}{r^2+a^2}+\frac{r\Delta}{(r^2+a^2)^2},\\
U_1 &= V+\frac{\Delta^2}{\beta}\Big[\Big(2\alpha+\frac{\beta_{,r}}{\Delta}\Big)_{,r}-\frac{\eta_{,r}}{\eta}\Big(\alpha+\frac{\beta_{,r}}{\Delta}\Big)\Big],\\
\alpha &= -i\frac{K\beta}{\Delta^2}+3iK_{,r}+\lambda+\frac{6\Delta}{r^2},\\
\beta &= 2\Delta\Big(-iK+r-M-\frac{2\Delta}{r}\Big).
\end{aligned} \tag{A.5}$$

B Functions of the source term

In this Appendix we show the A's in Eq.(3.19).

$$\begin{aligned}
A_{n\,n\,0} &= \frac{-2}{\sqrt{2\pi}\Delta^2}C_{n\,n}\rho^{-2}\overline{\rho}^{-1}L_1^+\{\rho^{-4}L_2^+(\rho^3S)\},\\
A_{\overline{m}\,n\,0} &= \frac{2}{\sqrt{\pi}\Delta}C_{\overline{m}\,n}\rho^{-3}\Big[\big(L_2^+S\big)\Big(\frac{iK}{\Delta}+\rho+\overline{\rho}\Big)\\
&\qquad -a\sin\theta S\frac{K}{\Delta}(\overline{\rho}-\rho)\Big],\\
A_{\overline{m}\,\overline{m}\,0} &= -\frac{1}{\sqrt{2\pi}}\rho^{-3}\overline{\rho}C_{\overline{m}\,\overline{m}}S\Big[-i\Big(\frac{K}{\Delta}\Big)_{,r}-\frac{K^2}{\Delta^2}+2i\rho\frac{K}{\Delta}\Big],\\
A_{\overline{m}\,n\,1} &= \frac{2}{\sqrt{\pi}\Delta}\rho^{-3}C_{\overline{m}\,n}[L_2^+S+ia\sin\theta(\overline{\rho}-\rho)S],\\
A_{\overline{m}\,\overline{m}\,1} &= -\frac{2}{\sqrt{2\pi}}\rho^{-3}\overline{\rho}C_{\overline{m}\,\overline{m}}S\Big(i\frac{K}{\Delta}+\rho\Big),\\
A_{\overline{m}\,\overline{m}\,2} &= -\frac{1}{\sqrt{2\pi}}\rho^{-3}\overline{\rho}C_{\overline{m}\,\overline{m}}S,
\end{aligned}$$

where S denotes ${}_{-2}S^{a\omega}_{\ell m}$.

C $\tilde{Z}_{\ell,m,n}$

In this appendix, the explicit forms of $\tilde{Z}_{\ell,m,n}$ which are used in this paper are shown.

$$\begin{aligned}
\frac{r_0^2}{\sqrt{2\pi}}\tilde{Z}_{2,2,0} &= \sqrt{\frac{128}{5}}\Bigg(v^4-\frac{107\,v^6}{42}+v^7\left(\frac{-17\,i}{3}+4\,i\,\gamma+2\,\pi-4\,q+4\,i\,\log(2)\right)\\
&+\left(-\frac{2173}{1512}+q^2\right)v^8\\
&+v^9\left(\frac{1819\,i}{126}-\frac{214\,i}{21}\gamma-\frac{107\,\pi}{21}+\frac{1145\,q}{189}-\frac{214\,i}{21}\log(2)\right)
\end{aligned}$$

$$+e^2\left(\frac{-13\,v^4}{2}+\frac{1493\,v^6}{84}+\left(-\frac{70915}{3024}-5\,q^2\right)\,v^8\right.$$
$$+v^7\left(\frac{136\,i}{3}-32\,i\,\gamma-16\,\pi+32\,q-32\,i\,\log(2)\right)$$
$$\left.\left.\left.+v^9\left(\frac{-9386\,i}{63}+\frac{2192\,i}{21}\gamma+\frac{1096\,\pi}{21}+\frac{2431\,q}{54}+\frac{2\,i}{r_0{}^2}+\frac{2192\,i}{21}\,\log(2)\right)\right)\right)\right)$$

$$\frac{r_0^2}{\sqrt{2\pi}}\tilde{Z}_{2,2,1} = \frac{-81}{\sqrt{10}}\,e\left(v^4-\frac{7\,v^6}{2}+v^7\left(\frac{-17\,i}{2}+6\,i\,\gamma+3\,\pi-\frac{14\,q}{3}+6\,i\,\log(2)\right)\right.$$
$$\left.+\left(2+q^2\right)\,v^8+v^9\left(\frac{153\,i}{4}-27\,i\,\gamma-\frac{27\,\pi}{2}+\frac{8\,q}{21}-27\,i\,\log(2)\right)\right)$$

$$\frac{r_0^2}{\sqrt{2\pi}}\tilde{Z}_{2,2,-1} = \frac{3}{\sqrt{10}}\,e\left(v^4+\frac{473\,v^6}{42}+v^7\left(\frac{-17\,i}{6}+2\,i\,\gamma+\pi-22\,q+2\,i\,\log(2)\right)\right.$$
$$+\left(\frac{36980}{567}+7\,q^2\right)\,v^8$$
$$\left.+v^9\left(\frac{-10183\,i}{252}+\frac{599\,i}{21}\gamma+\frac{599\,\pi}{42}-\frac{49450\,q}{189}+\frac{599\,i}{21}\,\log(2)\right)\right)$$

$$\frac{r_0^2}{\sqrt{2\pi}}\tilde{Z}_{2,1,0} = \frac{-i}{3}\sqrt{\frac{8}{5}}\left(v^5-\frac{3\,q\,v^6}{2}-\frac{17\,v^7}{28}\right.$$
$$+v^8\left(\frac{-10\,i}{3}+2\,i\,\gamma+\pi-\frac{439\,q}{126}+2\,i\,\log(2)\right)$$
$$+e^2\left(\frac{-11\,v^5}{2}+\frac{33\,q\,v^6}{4}+\frac{251\,v^7}{14}\right.$$
$$\left.\left.+v^8\left(\frac{271\,i}{12}-14\,i\,\gamma-7\,\pi-\frac{9041\,q}{504}-14\,i\,\log(2)\right)\right)\right)$$

$$\frac{r_0^2}{\sqrt{2\pi}}\tilde{Z}_{2,1,1} = \frac{i}{3}\sqrt{\frac{512}{5}}\,e\left(v^5-\frac{3\,q\,v^6}{2}-\frac{32\,v^7}{7}\right.$$
$$\left.+v^8\left(\frac{-20\,i}{3}+4\,i\,\gamma+2\,\pi+\frac{1055\,q}{252}+4\,i\,\log(2)\right)\right)$$

$$\frac{r_0^2}{\sqrt{2\pi}}\tilde{Z}_{2,0,1} = -\frac{e}{\sqrt{15}}\left(v^4-\frac{187\,v^6}{14}+v^7\left(\frac{-17\,i}{6}+2\,i\,\gamma+\pi+12\,q+2\,i\,\log(2)\right)\right.$$
$$+\left(\frac{3340}{63}-6\,q^2\right)\,v^8$$
$$\left.+v^9\left(\frac{3893\,i}{84}-\frac{229\,i}{7}\gamma-\frac{229\,\pi}{14}-\frac{1457\,q}{21}-\frac{229\,i}{7}\,\log(2)\right)\right)$$

$$\frac{r_0^2}{\sqrt{2\pi}}\tilde{Z}_{3,3,0} = \frac{i}{2}\sqrt{\frac{2187}{7}}\left(v^5-4\,v^7+v^8\left(\frac{-127\,i}{10}+6\,i\,\gamma+3\,\pi-5\,q+6\,i\,\log(2)\right)\right.$$
$$+e^2\left(\frac{-21\,v^5}{2}+\frac{115\,v^7}{4}\right.$$

$$+v^8\left(\frac{762\,i}{5}-72\,i\,\gamma-36\,\pi+\frac{171\,q}{2}-72\,i\,\log(2)\right)\Bigg)\Bigg)$$

$$\frac{r_0^2}{\sqrt{2\pi}}\tilde{Z}_{3,3,1} = \frac{-256\,i}{\sqrt{21}}\,e\left(v^5-\frac{173\,v^7}{36}\right.$$
$$\left.+v^8\left(\frac{-254\,i}{15}+8\,i\,\gamma+4\,\pi-6\,q+8\,i\,\log(2)\right)\right)$$

$$\frac{r_0^2}{\sqrt{2\pi}}\tilde{Z}_{3,3,-1} = 8\,i\,\sqrt{\frac{3}{7}}\,e\left(v^5+\frac{146\,v^7}{27}\right.$$
$$\left.+v^8\left(\frac{-127\,i}{15}+4\,i\,\gamma+2\,\pi-17\,q+4\,i\,\log(2)\right)\right)$$

$$\frac{r_0^2}{\sqrt{2\pi}}\tilde{Z}_{3,2,0} = \frac{i}{3}\sqrt{\frac{128}{7}}\left(v^6-\frac{4\,q\,v^7}{3}+e^2\left(-8\,v^6+\frac{32\,q\,v^7}{3}\right)\right)$$

$$\frac{r_0^2}{\sqrt{2\pi}}\tilde{Z}_{3,2,1} = \frac{-81\,i}{\sqrt{56}}\,e\left(v^6-\frac{4\,q\,v^7}{3}\right)$$

$$\frac{r_0^2}{\sqrt{2\pi}}\tilde{Z}_{3,2,-1} = \frac{i}{\sqrt{56}}\,e\left(v^6-\frac{4\,q\,v^7}{3}\right)$$

$$\frac{r_0^2}{\sqrt{2\pi}}\tilde{Z}_{3,1,0} = \frac{-i}{6\sqrt{35}}\left(v^5-\frac{8\,v^7}{3}+v^8\left(\frac{-127\,i}{30}+2\,i\,\gamma+\pi-\frac{77\,q}{9}+2\,i\,\log(2)\right)\right.$$
$$\left.+e^2\left(\frac{-5\,v^5}{2}+\frac{265\,v^7}{12}+v^8\left(\frac{254\,i}{15}-8\,i\,\gamma-4\,\pi+\frac{403\,q}{18}-8\,i\,\log(2)\right)\right)\right)$$

$$\frac{r_0^2}{\sqrt{2\pi}}\tilde{Z}_{3,1,1} = \frac{-8\,i}{3\sqrt{35}}\,e\left(v^5-13\,v^7\right.$$
$$\left.+v^8\left(\frac{-127\,i}{15}+4\,i\,\gamma+2\,\pi+\frac{131\,q}{9}+4\,i\,\log(2)\right)\right)$$

$$\frac{r_0^2}{\sqrt{2\pi}}\tilde{Z}_{3,0,1} = \frac{-i}{2\sqrt{105}}\,e\left(v^6-\frac{4\,q\,v^7}{3}\right)$$

$$\frac{r_0^2}{\sqrt{2\pi}}\tilde{Z}_{4,4,0} = \frac{1}{9}\sqrt{\frac{131072}{7}}\left(v^6-16\,e^2\,v^6\right)$$

$$\frac{r_0^2}{\sqrt{2\pi}}\tilde{Z}_{4,4,1} = \frac{-15625\,e\,v^6}{9\,\sqrt{224}}$$

$$\frac{r_0^2}{\sqrt{2\pi}}\tilde{Z}_{4,4,-1} = \frac{243\,e\,v^6}{\sqrt{224}}$$

$$\frac{r_0^2}{\sqrt{2\pi}}\tilde{Z}_{4,2,0} = \frac{-2^{5/2}}{31}\left(\frac{62\,v^6}{63}-\frac{248\,e^2\,v^6}{63}\right)$$

$$\frac{r_0^2}{\sqrt{2\pi}}\tilde{Z}_{4,2,-1} = \frac{-(e\,v^6)}{63\,\sqrt{2}}$$

$$\frac{r_0^2}{\sqrt{2\pi}}\tilde{Z}_{4,0,1} = \frac{e\,v^6}{84\,\sqrt{5}}$$

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